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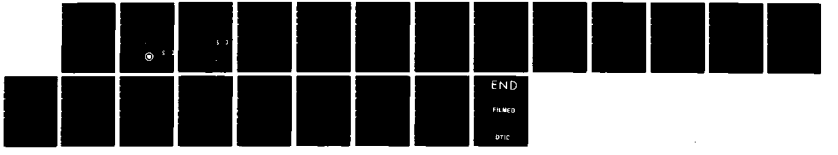

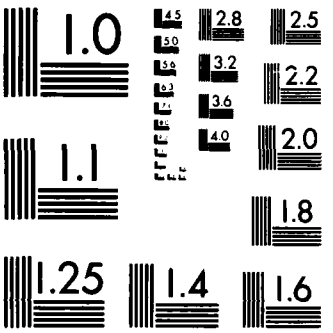

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by

N. Giri[1] and B. K. Sinha[2]

Center for Multivariate Analysis

University of Pittsburgh

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January 1985

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Center for Multivariate Analysis
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19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Let $X = (X_{ij}) = (\underset{\sim}{X}_1', \ldots, \underset{\sim}{X}_n')'$, $\underset{\sim}{X}_i' = (X_{i1}, \ldots, X_{ip})$ be a $n \times p$ random matrix with probability density function

$$f_X(x) = |\Sigma|^{-n/2} \, q(\operatorname{tr} \Sigma^{-1}(x - \underset{\sim}{e}\underset{\sim}{\mu}')'(x - \underset{\sim}{e}\underset{\sim}{\mu}'))$$

where $x \in \mathcal{X} = \{x: n \times p$ matrix $|$ rank of $n = p$, $\underset{\sim}{\mu} = (\mu_1, \ldots, \mu_p)' \in R^p$, $\underset{\sim}{e} = (1, \ldots, 1)'$ n-vector and $\Sigma > 0$ (positive definite). Assume that $q \in Q = \{q: [0,\infty) \to [0,\infty)$, convex$\}$ and $n > p$ so that $X'X > 0$ with probability one. It is proved that for

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testing H_0: $\underset{\sim}{\mu} = \underset{\sim}{0}$ versus the alternative H_1: $\underset{\sim}{\mu} \neq \underset{\sim}{0}$, the Hotelling's T^2-test is locally minimax, and for testing H_0: $\underset{\sim}{\mu}_{(1)} = \underset{\sim}{0}$ versus the alternative H_1: $\underset{\sim}{\mu}_{(1)} \neq \underset{\sim}{0}$, the appropriate Hotelling's T^2-test is both UMPI and locally minimax. In the second case, $\underset{\sim}{\mu}_{(1)} = (\mu_1,\ldots,\mu_{p_1})'$, $p_1 < p$ and $(\mu_{p_1},\ldots,\mu_p)$, Σ are unknown. The above results are those of Giri and Kiefer (Ann. Math. Statist., 1964) under the assumption q ~ normal. As a technical tool, Wijsman's representation theorem is used.

ROBUST TESTS OF MEAN VECTOR IN SYMMETRICAL MULTIVARIATE DISTRIBUTIONS

N. Giri and B. K. Sinha

ABSTRACT

Let $X = (X_{ij}) = (\underset{\sim}{X}_1', \ldots, \underset{\sim}{X}_n')'$, $\underset{\sim}{X}_i' = (X_{i1}, \ldots, X_{ip})$ be a $n \times p$ random matrix with probability density function

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Keywords and Phrases: Elliptically symmetric distributions, Hotelling's T^2-test, Hunt-Stein theorem, locally minimax tests, maximal invariant, Wijsman's representation theorem.

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ROBUST TESTS OF MEAN VECTOR IN SYMMERICAL MULTIVARIATE DISTRIBUTIONS

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0. Introduction and Summary

Let $X = (X_{ij}) = (\underset{\sim}{X}_1', \ldots, \underset{\sim}{X}_n')'$, $\underset{\sim}{X}_i' = (X_{i1}, \ldots, X_{ip})$ be a $n \times p$ random matrix with probability density function

$$f_X(x) = |\Sigma|^{-n/2} q(\text{tr } \Sigma^{-1}(x - \underset{\sim}{e}\underset{\sim}{\mu}')'(x - \underset{\sim}{e}\underset{\sim}{\mu}')) \qquad (1)$$

where $x \in X = \{x: n \times p \text{ matrix} \mid \text{rank of } x = p\}$, $\underset{\sim}{\mu} = (\mu_1, \ldots, \mu_p)' \in R^p$, $\underset{\sim}{e} = (1, \ldots, 1)'$ n-vector and $\Sigma > 0$ (positive definite $p \times p$). We shall assume that $q \in Q = \{q: [0,\infty) \to [0,\infty), \text{convex}\}$. This is a subclass of probability density functions which are left $O(n)$ ($n \times n$ orthogonal matrices) orthogonally invariant distributions about $\underset{\sim}{e}\underset{\sim}{\mu}'$ and is also a subclass of elliptically symmetric distributions about $\underset{\sim}{e}\underset{\sim}{\mu}'$ with scale matrix Σ. We shall assume throughout that $n > p$ so that $X'X > 0$ with probability one (see Giri (1977)).

We shall write for any p-vector $\underset{\sim}{b} = (b_1, \ldots, b_p)' = (\underset{\sim}{b}_{(1)}', \underset{\sim}{b}_{(2)}')$ with $\underset{\sim}{b}_{(1)} = (b_1, \ldots, b_{p_1})'$, $\underset{\sim}{b}_{(2)} = (b_{p_1+1}, \ldots, b_p)'$ and $\underset{\sim}{b}_{[i]} = (b_1, \ldots, b_i)'$ and for any

$p \times p$ matrix $A = (a_{ij}) = \begin{pmatrix} A_{(11)} & A_{(12)} \\ A_{(21)} & A_{(22)} \end{pmatrix}$ with $A_{(11)}, A_{(22)}$, $p_1 \times p_1$ and $p_2 \times p_2$ submatrices respectively, satisfying $p_1 + p_2 = p$. Also we shall write $A_{[i]} = \begin{pmatrix} a_{11}, \ldots, a_{1i} \\ a_{i1}, \ldots, a_{ii} \end{pmatrix}$ the $i \times i$ left-hand corner submatrix of A. Denote by

$$\underset{\sim}{\bar{X}} = \frac{1}{n} \sum_{1}^{n} \underset{\sim i}{X}, \qquad S = \sum_{1}^{n} (\underset{\sim i}{X} - \underset{\sim}{\bar{X}})(\underset{\sim i}{X} - \underset{\sim}{\bar{X}})'.$$

The assumption of multivariate normality i.e. when

$$q(\operatorname{tr} \Sigma^{-1}(x - e\underset{\sim}{\mu}')'(x - e\underset{\sim}{\mu}')) = (2\Pi)^{\frac{-np}{2}} \exp\{-\frac{1}{2} \operatorname{tr} \Sigma^{-1}(x - e\underset{\sim}{\mu}')'(x - e\underset{\sim}{\mu}')\} \qquad (2)$$

leads us to derive optimum test procedures for testing problems concerning $\underset{\sim}{\mu}$ and Σ. However it has been established that (see for example Kariya and Sinha (1984) and the references contained therein) optimum procedures can also be derived by replacing the multinormality assumption by one closely related to it, namely, the class of spherically symmetric distributions in the case of a random vector X ($p \times 1$) and the class of elliptically symmetric distributions in the case of an $n \times p$ random matrix X. These optimum procedures in turn involve the robustness of the distribution of the test statistic under the null hypothesis (null robustness), the robustness of the distribution of the test statistic under the alternatives (non-null robustness) and the robustness of the optimum properties of the test procedures (optimality robustness). We shall call a test robust in any one sense if an optimality property which the test enjoys can be extended to a class of distributions including the distribution under which the optimality holds.

We will consider here the following two testing problems about $\underset{\sim}{\mu}$ for the family of distributions given in (1).

(1) To test H_{10}: $\underset{\sim}{\mu} = \underset{\sim}{0}$ against the alternatives H_{11}: $\underset{\sim}{\mu} \neq \underset{\sim}{0}$ when Σ is unknown.

(2) To test H_{20}: $\underset{\sim}{\mu}_{(1)} = \underset{\sim}{0}$ against the alternatives H_{21}: $\underset{\sim}{\mu}_{(1)} \neq \underset{\sim}{0}$ when both $\underset{\sim}{\mu}_{(2)}$ and Σ are unknown.

Let $G_\ell(p)$ be the multiplicative group of p×p nonsingular matrices g. The first problem remains invariant under $G_\ell(p)$ with the action

$$(\bar{\underset{\sim}{X}}, S; \underset{\sim}{\mu}, \Sigma) \to (g\bar{\underset{\sim}{X}}, gSg'; g\underset{\sim}{\mu}, g\Sigma g'), \quad g \in G_\ell(p).$$

A maximal invariant in the space $(\bar{\underset{\sim}{X}}, S)$ under $G_\ell(p)$ is $T^2 = n\bar{\underset{\sim}{X}}' S^{-1} \bar{\underset{\sim}{X}}$ or equivalently $R^2 = n\bar{\underset{\sim}{X}}'(S + n\bar{\underset{\sim}{X}}\bar{\underset{\sim}{X}}')^{-1}\bar{\underset{\sim}{X}} = T^2/(1+T^2)$. A corresponding maximal invariant in the parametric space of $(\underset{\sim}{\mu}, \Sigma)$ is $\delta^2 = n\underset{\sim}{\mu}'\Sigma^{-1}\underset{\sim}{\mu}$. Kariya (1981) has proved that the Hotelling's T^2 test which rejects H_{10} whenever $T^2 \geq C$ or equivalently $R^2 \geq C$, where C is a constant depending on the level α of the test, is uniformly most powerful invariant (UMPI), whatever $q \in Q$. The distribution of T^2 under H_{10} is the same as that of T^2 under the multivariate normal set-up (Kariya, 1982). Needless to mention that the multivariate normal distribution belongs to the family given in (1). We shall show in section 1 that the Hotelling's T^2 test is locally minimax in the sense of Giri and Kiefer (1964) for testing H_{10} against H_{11} in (1) as $\delta^2 \to 0$. In the multivariate normal setup, this result is proved in Giri and Kiefer (1984).

Let T_1 be the group of translations such that $t_1 \in T_1$ translates the last p_2 components of each $\underset{\sim}{X}_i$, i = 1,...,n and let G be the multiplicative group of p×p nonsingular matrices of the form

$$g = \begin{pmatrix} g_{(11)} & 0 \\ g_{(21)} & g_{(22)} \end{pmatrix} \tag{3}$$

where $g_{(11)}$ is the upper left hand corner $p_1 \times p_1$ submatrix of g. The second problem remains invariant under the affine group (G,T_1) such that for $g \in G$, $t_1 \in T_1$

$$(g,t_1)\underset{\sim}{X}_i = g\underset{\sim}{X}_i + \underset{\sim}{t}_1, \quad i = 1,\ldots,n.$$

A maximal invariant under (G,T_1) is given by (see Giri (1977))

$$\begin{aligned} \bar{R}_1^2 &= n\underset{\sim}{\bar{X}}'_{(1)}(S_{(11)} + r\underset{\sim}{\bar{X}}_{(1)}\underset{\sim}{\bar{X}}'_{(1)})^{-1}\underset{\sim}{\bar{X}}_{(1)} \\ &= n\underset{\sim}{\bar{X}}'_{(1)}S^{-1}_{(11)}\underset{\sim}{\bar{X}}_{(11)}/(1 + n\underset{\sim}{\bar{X}}'_{(1)}S^{-1}_{(11)}\underset{\sim}{\bar{X}}_{(1)}) \end{aligned} \tag{5}$$

A corresponding maximal invariant in the parametric space, under the induced group of transformations, is

$$\bar{\delta}_1^2 = n\underset{\sim}{\mu}'_{(1)}\Sigma^{-1}_{(11)}\underset{\sim}{\mu}_{(1)}. \tag{6}$$

For any invariant test under (G,T_1), the second problem reduces to testing H_{20}: $\bar{\delta}_1^2 = 0$ against the alternatives H_{21}: $\bar{\delta}_1^2 > 0$. We shall show in section 2 that the test which rejects H_{20} whenever $\bar{R}_1^2 \geq C$ is UMPI and locally minimax as $\bar{\delta}_1^2 \to 0$ for (1). In the multivariate normal setup this test has been proved to be UMPI and locally minimax as $\bar{\delta}_1^2 \to 0$ (Giri and Kiefer (1964)).

1. Locally minimax test for problem 1

The theory of locally minimax tests has been developed in Giri and Kiefer (1964). We refer to this paper for details. For each (δ^2,η) in the parametric space Ω, where $\delta^2 > 0$ and η is of arbitrary dimension and its range may depend on δ^2, let $p(x; \delta^2,\eta)$ be a probability density function on (X,A) with respect

to some σ-finite measure. Suppose that we are interested in testing at level α $(0 < \alpha < 1)$ the hypothesis H_{10}: $\delta^2 = 0$ against the alternatives H_{11}^*: $\delta^2 = \lambda$ where λ is a positive constant. For fixed α, consider the critical region of the form

$$\Delta = \{x: U(x) \geq C_\alpha\} \tag{7}$$

where U is bounded and positive and has a continuous distribution function for each (δ^2,η), equicontinuous in (δ^2,η) for some $\delta^2 < \delta_0^2$ and that

$$P_{0,\eta}(\Delta) = \alpha \tag{8}$$

$$P_{\lambda,\eta}(\Delta) = \alpha + h(\lambda) + g(\lambda,\eta) \tag{9}$$

where $g(\lambda,\eta) = o(h(\lambda))$ uniformly in η with $h(\lambda) > 0$ for $\lambda > 0$ and $h(\lambda) = 0(1)$. Let $\xi_{0,\lambda}$ $\xi_{1,\lambda}$ denote the a priori probability density function on the sets $\{\delta^2 = 0\}$, $\{\delta^2 = \lambda\}$ respectively such that

$$\frac{\int p(x:\lambda,\eta)\xi_{1\lambda}(d\eta)}{\int p(x:\ 0,\eta)\xi_{0\lambda}(d\eta)} = 1 + h(\lambda)[g(\lambda) + r(\lambda)U(x)] + B(x,\lambda) \tag{10}$$

where $0 < c_1 < r(\lambda) < c_2 < \infty$ for λ sufficiently small and $g(\lambda) = o(1)$ and $B(x,\lambda) = o(h(\lambda))$ uniformly in x. If U satisfies (8) and (9) and if for sufficiently small λ there exist $\xi_{0\lambda}$ and $\xi_{1\lambda}$ satisfying (10) then Δ is locally minimax for testing H_{10}: $\delta^2 = 0$ against the alternatives H_{11}': $\delta^2 = \lambda$ (specified) as $\lambda \to 0$.

It is wellknown that (see for example Giri, Kiefer and Stein (1963), Giri, Kiefer (1964)) the Hunt-Stein Theorem cannot be applied for the group $G_2(p)$ with $p \geq 2$. However this theorem does apply for the smaller group

$$G_T = \{p\times p \text{ nonsingular lower triangular matrices}\}$$

with $p \geq 2$. Thus for each δ^2, there is a level α test which is invariant under G_T (see Lehmann (1959), p. 225) and which minimizes among all level α tests, the minimum power under H_{11}'. In the place of R^2 under $G_\ell(p)$, we obtain a p-dimensional vector $\underset{\sim}{R} = (R_1^2,\ldots,R_p^2)'$ as a maximal invariant under G_T and $\underset{\sim}{R}$ is defined by

$$\sum_{j=1}^{i} R_j^2 = n\, \bar{\underset{\sim}{X}}_{[i]}'(S_{[ii]} + n\, \bar{\underset{\sim}{X}}_{[i]}\bar{\underset{\sim}{X}}_{[i]}')^{-1}\bar{\underset{\sim}{X}}_{[i]} \qquad (11)$$

$$i = 1,\ldots,p$$

with $R_i^2 \geq 0$ and $\sum_1^p R_i^2 = R^2$. A corresponding maximal invariant in the parametric space under the induced group is $\underset{\sim}{\delta} = (\delta_1^2,\ldots,\delta_p^2)'$ and $\delta_1^2,\ldots,\delta_p^2$ are given by

$$\sum_1^i \delta_j^2 = n\underset{\sim}{\mu}_{[i]}'\Sigma_{[ii]}^{-1}\underset{\sim}{\mu}_{[i]}, \quad i = 1,\ldots,p \qquad (12)$$

with $\delta_i^2 \geq 0$, $\sum_1^p \delta_i^2 = \delta^2$. The nuisance parameter in this reduced setup is

$$\underset{\sim}{\eta} = (\frac{\delta_1^2}{\delta^2},\ldots,\frac{\delta_p^2}{\delta^2})' = (\eta_1,\ldots,\eta_p)'. \qquad (13)$$

Since the distribution of $\underset{\sim}{R}$ depends on Ω only through $\underset{\sim}{\delta}$ we may put $\Sigma = I$ and redefine $\sqrt{n}\,\underset{\sim}{\mu} = \underset{\sim}{\alpha} = (\delta_1,\ldots,\delta_p)'$. Using Stein's theorem (1956) or Wijsman's representation theorem (1967), the ratio of the probability densities of $\underset{\sim}{R}$, under H_{11}' and under H_{10}, is given by (for $g \in G_T$)

$$\text{ratio} = \frac{\int_{G_T} p_1(g\underset{\sim}{\bar{x}}, gsg')\prod_1^p (g_{ii}^2)^{\frac{n-i}{2}}\, dg}{\int_{G_T} p_0(g\underset{\sim}{\bar{x}}, gsg')\prod_1^p (g_{ii}^2)^{\frac{n-i}{2}}\, dg} \qquad (14)$$

where

$$p_1(\bar{x},s) = |s|^{\frac{n}{2}} q(tr\{(s+n\bar{x}\bar{x}') - 2n\bar{x}'\mu+\delta^2\})$$

$$p_0(\bar{x},s) = |s|^{\frac{n}{2}} q(tr((s+n\bar{x}\bar{x}')).$$

Let A be a matrix belonging to G_T such that $A(s+n\bar{x}\bar{x}')A' = I_p$. Then $A'A = (s+n\bar{x}\bar{x}')^{-1} = s^{-1} - ns^{-1}\bar{x}\bar{x}'s^{-1}(1 + n\bar{x}'s^{-1}\bar{x})^{-1}$, so that $n\bar{x}'A'A\bar{x} = n\bar{x}'s^{-1}\bar{x}(1 + n\bar{x}'s^{-1}\bar{x})^{-1} = \sum_1^p R_j^2 = R^2$. Since $A_{[ii]}(s_{[ii]} + n\bar{x}_{[i]}\bar{x}'_{[i]})A'_{[ii]} = I_i$, we obtain $n\bar{x}'_{[i]}A'_{[ii]}A_{[ii]}\bar{x}_{[i]} = \sum_1^i R_j^2$, so that

$$\sqrt{n}\, A\bar{x} = (R_1,\ldots,R_p)' = y \tag{15}$$

where $y'_{[i]}y_{[i]} = \sum_1^i R_j^2$. Writing $gA^{-1} = g$ we get from (14)

$$\text{ratio} = \frac{\int_{G_T} q(tr(gg' - 2\,\alpha'gy + \delta^2)) \prod_{i=1}^p (g_{ii}^2)^{\frac{n-i}{2}} dg}{\int_{G_T} q(tr(gg') \prod_1^p (g_{ii}^2)^{\frac{n-i}{2}} dg} \tag{16}$$

where $\alpha = (\delta_1,\ldots,\delta_p)'$.

Let us assume that q is thrice continuously differentiable. Writing $q^{(i)}(x) = \frac{d^i q}{dx^i}$ we obtain

$$\begin{aligned} &q(tr(gg' - 2\,\alpha'gy + \delta^2)) \\ &= q(tr\,gg') + q^{(1)}(tr\,gg')(-2tr(\alpha'gy)+\lambda) \\ &+ \frac{q^{(2)}}{2}(tr\,gg')(-2\,tr\,(\alpha'gy)+\lambda)^2 \\ &+ \frac{1}{6}\,q^{(3)}(Z)(-2\,tr\,(\alpha'gy)+\lambda)^3 \end{aligned} \tag{17}$$

where $Z = tr\,gg' + (1-\alpha)(-2\,tr\,(\alpha'gy)+\lambda)$, $\quad 0 \le \alpha \le 1$. Let

$$D = \int_{G_T} q(tr\,gg') \prod_1^p (g_{ii}^2)^{\frac{n-i}{2}} dg. \tag{18}$$

Since the measures $q^{(i)}(\text{tr } gg') \prod_1^p (g_{ii}^2)^{\frac{n-i}{2}} dg$, $i = 1,2,3$ are invariant under the change of sign of g to -g, we get

$$\int_{G_T} \text{tr}(\underline{\alpha}' g \underline{y}) q^{(i)}(\text{tr } gg') \prod_{i=1}^p (g_{ii}^2)^{\frac{n-i}{2}} dg = 0 \tag{19}$$

and

$$\int_{G_T} g_{ij} g_{\ell k} q^{(i)}(\text{tr } gg') \prod_{i=1}^p (g_{ii}^2)^{\frac{n-i}{2}} dg = 0 \tag{20}$$

$$\text{if } i \neq \ell,\ j \neq k.$$

Thus from (16)-(20) we get

$$\begin{aligned} \text{ratio} = 1 &+ \frac{\lambda}{D} \int_{G_T} q^{(1)}(\text{tr } gg') \prod_1^p (g_{ii}^2)^{\frac{n-i}{2}} dg \\ &+ \frac{2}{D} \int_{G_T} (\text{tr } \underline{\alpha}' g \underline{y})^2 q^{(2)}(\text{tr } gg') \prod_1^p (g_{ii}^2)^{\frac{n-i}{2}} dg \\ &+ \frac{\delta^4}{2D} \int_{G_T} q^{(2)}(\text{tr } gg') \prod_1^p (g_{ii}^2)^{\frac{n-i}{2}} dg \\ &+ \frac{1}{6D} \int (-2 \text{ tr } \underset{\sim}{\alpha}' g \underset{\sim}{y} + \lambda)^3 \, q^{(3)}(z) \prod_1^p (g_{ii}^2)^{\frac{n-i}{2}} dg. \end{aligned} \tag{21}$$

The first integral in (21) is a finite constant β_1. To evaluate the second integral in (21) we first note that

$$\text{tr } \underline{\alpha}' g \underline{y} = \sum_j r_j [\sum_{i>j} \delta_i g_{ij}^2 + \delta_j g_{jj}]. \tag{22}$$

From (20) and (22) the second integral can be written as

$$\int_{G_T} (\sum_i r_j^2 \sum_{i>j} \delta_i^2 g_{ij}^2 + \delta_j^2 g_{jj}^2) q^{(2)}(\text{tr } gg') \prod_1^p (g_{ii}^2)^{\frac{n-i}{2}} dg. \tag{23}$$

Let us now evaluate the integral

$$I = \int_{G_T} g_{ii}^2 q^{(2)}(\text{tr } gg') \prod_1^p (g_{ii}^2)^{\frac{n-i}{2}} dg. \tag{24}$$

Define

$$\begin{aligned} L &= \text{tr}(gg'), \\ e_i &= g_{ii}^2/L \ , \ i = 1,2,\ldots,p \ ; \\ e_{p+i} &= g_{i+1,i}^2/L, \ i = 1,\ldots,p-1 \ ; \\ e_{p+p-1+i} &= g_{i+2,i}^2, \ i = 2,\ldots,p-2 \\ &\cdots\cdots \\ e_{\frac{p(p+1)}{2}} &= g_{p1}^2/L. \end{aligned} \tag{25}$$

Write

$$\begin{aligned} K &= \int_{G_T} q^{(2)}(\text{tr } gg')dg \\ N &= \int L^{\sum_1^p (\frac{n-i}{2})} q^{(2)}(L)dg. \end{aligned} \tag{26}$$

Since $\frac{q^{(2)}(\text{tr } gg')}{K}$ is a spherical density of g_{ij}'s, L and $\underset{\sim}{e} = (e_1,\ldots,e_{\frac{p(p+1)}{2}})'$ are independent and $\underset{\sim}{e}$ obeys a Dirichlet distribution $D(\frac{1}{2},\ldots,\frac{1}{2})$. From Kariya and Eaton (1977) the probability density function of $\underset{\sim}{e}$ is given by

$$p(\underset{\sim}{e}) = \frac{\Gamma(\frac{p(p+1)}{2})}{[\Gamma(\frac{1}{2})]^{p(p+1)/2}} \prod_1^{\frac{p(p+1)}{2}} (e_i)^{\frac{1}{2}-1} (1 - \sum_1^{\frac{p(p+1)}{2}} e_i)^{\frac{1}{2}-1}.$$

Now using (24)-(26) we get

$$\begin{aligned} I &= \frac{N}{K} E(\prod_{j\neq i} (e_j)^{\frac{n-j}{2}} e_i^{\frac{n-i+2}{2}}) \\ &= \frac{N}{K} M_{\frac{(n-i+1)}{2}}, \end{aligned} \tag{27}$$

where

$$M = E(\prod_{1}^{p}(e_i)^{\frac{n-i}{2}}).$$

Similarly

$$\int_{G_T} g_{ij}^2 q^{(2)}(\text{tr } gg')\prod_{1}^{p}(g_{ii}^2)^{\frac{n-i}{2}} dg$$

$$= E(e_k \prod_{1}^{p} e_i^{\frac{n-i}{2}}) = \frac{NM}{2k} \text{ with } k > p. \tag{28}$$

It may be verified that the third and the fourth integrals in (21) are both of the order $o(\delta^2)$ uniformly in $\underline{y}$ and $\underline{\eta}$. Hence we can write

$$\text{ratio} = 1 + \frac{\lambda}{D}\left(\beta_1 + \frac{MN}{K}\left(\sum_{1}^{p} r_j^2\left(\sum_{i>j} \eta_i^2 + (n-j+1)\eta_j^2\right)\right)\right.$$

$$\left. + B(\underline{y},\underline{\eta},\lambda)\right) \tag{29}$$

where $B(\underline{y},\underline{\eta},\lambda) = o(\lambda)$ uniformly in $\underline{y}$ and $\underline{\eta}$. Thus, from (29), the equation (10) is satisfied by letting $\xi_{0\lambda}$ give measure one to the single point $\underline{\eta} = \underline{0}$ while $\xi_{1\lambda}$ give measure one to the single point $\underline{\eta} = \underline{\eta}^* = (\eta_1^*,\ldots,\eta_p^*)'$ whose jth coordinate η_j^* satisfies

$$\eta_j^* = (n-j)^{-1}(n-j+1)^{-1}p^{-1}n(n-p), \quad j = 1,\ldots,p$$

so that

$$\sum_{i>j} \eta_i^{*2} + (n-j+1)\eta_j^{*2} = \frac{n}{p}. \tag{30}$$

The equation (8) follows from the null robustness of the distribution of T^2 (Kariya (1982)). To show (9) we proceed as follows.

For $g \in G_\ell$, we can write using Wijsman's theorem

$$\frac{f_{T^2}(t^2|H'_{11})}{f_{T^2}(t^2|H_{10})} = \frac{\int_{G_\ell} q(\mathrm{tr}(gg' - 2\underline{\alpha}'g\underline{y}+\lambda)|gg'|^{\frac{n-p}{2}}\,dg}{\int_{G_\ell} q(\mathrm{tr}\ gg')|gg'|^{\frac{n-p}{2}}\,dg} \tag{31}$$

To evaluate (31) we use the following results.

(i) Given $\underline{x} = (x_1,\ldots,x_p)'$ there exists an $0 \in O(p)$, the group of $p\times p$ orthogonal matrices, such that

$$0\underline{x} = (\sqrt{\underline{x}'\underline{x}}\ ,\ 0,\ldots,0)'. \tag{32}$$

(ii) We can decompose $G_\ell = G_T \times O(p)$. Denote

$$\mu(dg) = |gg'|^{\frac{n-p}{2}}\,dg,\ \xi(dh) = \prod_1^p (h_{ii}^2)^{\frac{n-i}{2}}\,dh \tag{33}$$

for $h \in G_T$ and $\tau(d0)$ the invariant measure on $O(p)$. Then for $g \in G_\ell$, $\mu(dg) = \xi(dh)\ \tau(d0)$.

(iii) $$\int_{O(p)} \mathrm{tr}(AOBO')\tau(d0) = \frac{\mathrm{tr}AB}{p}$$

$$\text{and } \int_{O(p)} (\mathrm{tr}\ OA)^k \tau(d0) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \dfrac{\mathrm{tr}A'A}{p} & \text{if } k = 2. \end{cases} \tag{34}$$

Using the above results we can rewrite (31) as

$$\begin{aligned} &1 + \frac{\lambda}{D}\int_{G_T} q^{(1)}(\mathrm{tr}\ hh')\prod_1^p (h_{ii}^2)^{\frac{n-i}{2}}\,dh \\ &+ \frac{2\lambda r^2}{p^2 D}\int_{G_T} h_{11}^2 q^{(2)}(\mathrm{tr}\ hh')\prod_1^p (h_{ii}^2)^{\frac{n-i}{2}}\,dh + B(\underline{y},\underline{\eta},\lambda) \\ &= 1 + \lambda(K + cr^2) + B(\underline{y},\underline{\eta},\lambda) \end{aligned} \tag{3.5}$$

where $B(\underset{\sim}{y},\underset{\sim}{\eta},\lambda) = o(\lambda)$ uniformly in $\underset{\sim}{y}$ and $\underset{\sim}{\eta}$ and k,c are positive constants (using (27) and (28)). Hence for T^2-test, with $b > 0$,

$$P_{\lambda,\eta}^{(\Delta)} = o + b\lambda + g(\lambda,\underset{\sim}{\eta})$$

with $g(\lambda,\underset{\sim}{\eta}) = o(b\lambda)$.

Hence we get the following theorem.

Theorem 1. For testing H_{10} against H_{11}', Hotelling's T^2 test is locally minimax as $\lambda \to 0$ for the family of distributions given in (1).

2. UMP invariant and locally minimax test for problem 2

Write $\underset{\sim}{X}_i = \begin{pmatrix} \underset{\sim}{X}_{i(1)} \\ \underset{\sim}{Y}_{i(2)} \end{pmatrix}$, $i = 1,\ldots,n$ and $X = (X_1\ X_2)$ with $X_1 = \begin{pmatrix} \underset{\sim}{Y}_{1(1)} \\ \underset{\sim}{X}_{n(1)} \end{pmatrix}$, $X_2 = \begin{pmatrix} \underset{\sim}{X}_{1(2)} \\ \underset{\sim}{X}_{n(2)} \end{pmatrix}$ with X_1: $n \times p_1$, X_2: $n \times p_2$. Now writing $\Sigma_{22.1} = \Sigma_{(22)} - \Sigma_{(21)}\Sigma_{(11)}^{-1}\Sigma_{(12)}$, $\underset{\sim}{e} = (\underset{\sim}{e}_{(1)}', \underset{\sim}{e}_{(2)}')'$, we can write

$$f_X(x) = |\Sigma_{(11)}|^{-n/2}|\Sigma_{22.1}|^{-n/2}$$

$$\cdot\ q(\mathrm{tr}\{\Sigma_{(11)}^{-1}(\underset{\sim}{x}_1 - \underset{\sim}{e}_{(1)}\underset{\sim}{\mu}_{(1)}')'(\underset{\sim}{x}_1 - \underset{\sim}{e}_{(1)}\underset{\sim}{\mu}_{(1)}') + \Sigma_{22.1}^{-1}u'u\})$$

where

$$u = X_2 - \underset{\sim}{e}_2\underset{\sim}{\mu}_{(2)}' - (\underset{\sim}{x}_1 - \underset{\sim}{e}_{(1)}\underset{\sim}{\mu}_{(1)}')\Sigma_{(11)}^{-1}\Sigma_{(12)}.$$

The marginal probability density function of X_1 is given by

$$f_{X_1}(x_1) = |\Sigma_{(11)}|^{-n/2}\tilde{q}(\mathrm{tr}\ \Sigma_{(11)}^{-1}(x_1 - e_{(1)}\mu_{(1)}')'(x_1 - e_{(1)}\mu_{(1)}')) \qquad (37)$$

where, taking $w = \Sigma_{22.1}^{-1/2}u$,

$$\tilde{q}(\text{tr } \Sigma_{(11)}^{-1}V) = \int_{R^{p_2 n}} q(\text{tr}(\Sigma_{(11)}^{-1}V + ww'))dw.$$

The assumed convexity of q implies convexity of $\tilde{q}$.

To find the ratio of the probability densities of the maximal invariant $\bar{R}_1^2$ with respect to the group (G,T_1) we can without any loss of generality consider only the multiplicative group $G_\ell(p_1)$which transforms each $\underset{\sim}{X}_{i(1)} \to g\underset{\sim}{X}_{i(1)}$, $g \in G_\ell(p_1)$.

Using Stein's theorem or the Wijsman's representation theorem we obtain for $g \in G_\ell(p_1)$, the ratio of probability densities of $\bar{R}_1^2$ under H_{21} and H_{20} as

$$\text{ratio} = \frac{\int_{G_\ell(p_1)} \tilde{q}(\text{tr}(gg' - 2\underline{\rho}'g\underline{y} + \bar{\delta}_1^2))|gg'|^{\frac{n-p_1}{2}} dg}{\int_{G_\ell(p_1)} \tilde{q}(\text{tr } gg')|gg'|^{\frac{n-p_1}{2}} dg} \tag{38}$$

where $\underline{\rho} = (\rho_1,\ldots,\rho_{p_1})'$, $\underline{y} = (r_1,\ldots,r_{p_1})'$ satisfying $\underline{\rho}'\underline{\rho} = \bar{\delta}_1^2$, $\underline{y}'\underline{y} = \bar{r}_1^2$. Using (32) we can write the numerator of (38) as

$$\int \tilde{q}(\text{tr}(gg' - 2\underline{\rho}^{*\prime}g\underline{y}^* + \bar{\delta}_1^2)|gg'|^{\frac{n-p_1}{2}} dg = H(\bar{r}_1) \text{ (say)}$$

with $\underline{\rho}^* = (\bar{\delta}_1,0,\ldots,0)'$, $\underline{y}^* = (\bar{r}_1,0,\ldots,0)'$. Since $|gg'|^{\frac{n-p_1}{2}} dg$ is invariant under the sign change, $g \to -g = h \in G_\ell(p_1)$ we conclude that

$$H(\bar{r}_1) = \int_{G_\ell(p_1)} \tilde{q}(\text{tr}(gg' - 2\underline{\rho}^{*\prime}g(-\underline{y}^*) + \bar{\delta}_1^2))|gg'|^{\frac{n-p_1}{2}} dg = H(-\bar{r}_1). \tag{39}$$

Hence $H(\bar{r}_1) = \alpha H(\bar{r}_1) + (1-\alpha)H(-\bar{r}_1)$. Since $\tilde{q}$ is convex by assumption, for

$\frac{1}{2} < \alpha < 1$,

$$H(\bar{r}_1) \geq H((2\alpha-1)\bar{r}_1). \tag{40}$$

From (39)-(40), we conclude that $H(\bar{r}_1)$ is a monotonically increasing funciton of $\bar{r}_1^2$. Now applying Neyman-Pearson lemma we obtain the following theorem.

Theorem 2. For testing H_{20}: $\bar{\delta}_1^2 = 0$ against the alternative H_{21}: $\bar{\delta}_1^2 > 0$ the test which rejects H_{20} whenever $\bar{R}_1^2 \geq c$ is UMPI with respect to the group (G,T_1) for the family of distributions given in (1).

Moreover, we can prove in an analogous way, using the results of section 1, the following theorem. The details are omitted.

Theorem 3. For testing H_{20} against H_{21}: $\bar{\delta}_1^2 = \lambda$ (specified) > 0, Hotelling's T^2 test which rejects H_{20} whenever $\bar{R}_1^2 \geq c$ is locally minimax as $\lambda \to 0$.

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